

Topological geon black holes in Einstein-Yang-Mills theory

George T. Kottanattu* and Jorma Louko†

*School of Mathematical Sciences, University of Nottingham,
Nottingham NG7 2RD, UK*

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Abstract

We construct topological geon quotients of two families of Einstein-Yang-Mills black holes. For Künzle's static, spherically symmetric $SU(n)$ black holes with $n > 2$, a geon quotient exists but generically requires promoting charge conjugation into a gauge symmetry. For Kleihaus and Kunz's static, axially symmetric $SU(2)$ black holes a geon quotient exists without gauging charge conjugation, and the parity of the gauge field winding number determines whether the geon gauge bundle is trivial. The geon's gauge bundle structure is expected to have an imprint in the Hawking-Unruh effect for quantum fields that couple to the background gauge field.

*george.kottanattu@alumni.nottingham.ac.uk

†jorma.louko@nottingham.ac.uk

1 Introduction

Given a stationary black hole spacetime with a bifurcate Killing horizon, it may be possible to construct from it a time-orientable quotient spacetime in which the two exterior regions separated by the Killing horizon become identified. If the quotient is asymptotically flat, its spatial geometry is that of a compact manifold minus a point, with the omitted point at an asymptotically flat infinity. This makes the quotient a topological geon in the sense introduced by Sorkin [1], as motivated by the earlier work in [2, 3, 4]. The showcase example is the \mathbb{RP}^3 geon [5, 6, 7, 8], formed as a \mathbb{Z}_2 quotient of Kruskal. There exist also quotients in which the infinity is only asymptotically locally flat, and others in which the infinity is asymptotically anti-de Sitter or asymptotically locally anti-de Sitter [9, 10, 11, 12]. In this paper we shall understand topological geons to encompass all of these cases, the characteristic property being that the infinity consists of only one component.

Topological geon black holes of the kind described above are unlikely to be created in an astrophysical star collapse, as their formation from conventional initial data would require a change in the spatial topology. However, they provide an arena for the Hawking-Unruh effect in a setting where the black hole is eternal and has nonvanishing surface gravity but thermality for a quantum field cannot arise by the usual procedure of tracing over a causally disconnected exterior [13]. There is still thermality, in the usual Hawking temperature, but only for a limited set of observations, and the non-thermal correlations bear an imprint of the absence of the causally disconnected exterior [9, 10, 11, 12, 14, 15]. In a sense, the Hawking-Unruh effect on a topological geon black hole reveals to an exterior observer features of the geometry that are classically hidden behind the horizons. A recent review can be found in [16].

When the black hole has a Maxwell field, it may be necessary to include charge conjugation in the map with which the black hole gauge bundle is quotiented into the geon gauge bundle [12]. This happens for example for the Reissner-Nordström hole, both with electric and magnetic charge; it also happens for the higher-dimensional Reissner-Nordström hole with electric charge in any dimension and with magnetic charge in even dimensions. Maxwell's theory on the geon must then incorporate charge conjugation as a gauge symmetry, rather than just as a global symmetry: technically, the gauge group is no longer $U(1) \simeq SO(2)$ but $\mathbb{Z}_2 \ltimes U(1) \simeq O(2)$, where the nontrivial element of \mathbb{Z}_2 acts on $U(1)$ by complex conjugation [17]. The presence of the charge conjugation in the quotienting map can further be verified to leave its imprint in the Hawking-Unruh effect for a quantum field that couples to the background Maxwell field [18, 19]. By contrast, spherically symmetric Einstein-SU(2) black holes admit a geon quotient without the inclusion of charge conjugation in the quotienting map, and the geon's gauge bundle is in fact trivial [12].

The purpose of this paper is to construct two new families of Einstein-Yang-Mills geon black holes. We shall specifically examine whether charge conjugation needs to be promoted into a gauge symmetry when taking the geon quotient. We take the gauge group to be $SU(n)$ with $n \geq 2$, a choice motivated physically by the appearance of

these groups in particle physics and mathematically by their amenability to a unified treatment. We shall work with pure Einstein-Yang-Mills, but we note that these gauge groups, and the definition of spherical symmetry in terms of $SU(2)$ rather than $SO(3)$ [20, 21], provide opportunities for extensions that include spinor as well as scalar fields.

In Sections 2 and 3 we consider the static, spherically symmetric Einstein- $SU(n)$ black holes of Künzle [22] and their generalisations to a negative cosmological constant [23, 24]. The case $n = 2$ was covered in [12] as discussed above. For $n > 2$ we show that a geon quotient exists and generically requires including charge conjugation in the quotienting map: the enlarged gauge group is $\mathbb{Z}_2 \times SU(n)$, where the nontrivial element of \mathbb{Z}_2 acts on $SU(n)$ by complex conjugation. A quotient without charge conjugation is possible only for certain special field configurations, of which we give a complete list, and we show that the geon gauge bundle is then trivial.

In Sections 4 and 5 we consider the static, axially symmetric Einstein- $SU(2)$ black holes of Kleihaus and Kunz [25, 26]. We show that all these holes admit a geon quotient without gauging charge conjugation. When the winding number of the gauge field configuration is odd, the geon gauge bundle is trivial; this includes as a special case the spherically symmetric geon discussed in [12]. When the winding number is even, the geon gauge bundle is nontrivial.

Section 6 summarises the results and discusses their relevance for the Hawking-Unruh effect.

The metric signature is $(-+++)$. Sections 2 and 3 use the convention of an antihermitian gauge field, common in mathematical literature. Sections 4 and 5 use the convention of a hermitian gauge field, common in physics literature. Homotopies are assumed smooth, without loss of generality [27].

2 Spherically symmetric $SU(n)$ black holes

In this section we review the relevant properties of the static, spherically symmetric $SU(n)$ Einstein-Yang-Mills black holes of Künzle [22] and their generalisations to a negative cosmological constant [23, 24]. We also give explicit Kruskal-like coordinates that extend these solutions across the Killing horizon. We assume $n > 2$ when not explicitly mentioned otherwise, although most of the formulas hold also for $n = 2$.

2.1 Exterior ansatz

Given a group action on the base space of a principal bundle, the notion of a symmetric gauge field can be formulated as invariance of the connection under an appropriate group action on the total space. Concretely, let P be a principal bundle with base manifold \mathcal{M} , projection $\pi : P \rightarrow \mathcal{M}$ and structure group G with Lie algebra \mathfrak{g} . Let H be a group and $\phi : H \times \mathcal{M} \rightarrow \mathcal{M}$ its action on \mathcal{M} . We say that the connection form $\omega \in \Lambda^1(P, \mathfrak{g})$ is H -symmetric if the following three conditions hold:

1. For each $h \in H$, there is a $\Phi_h \in \text{Aut}(P)$ such that $\pi \circ \Phi_h = \phi_h \circ \pi$ with $\Phi_{\text{Id}_H} = \text{Id}_P$;

2. $\Phi_h^* \omega = \omega$ for all $h \in H$;
3. The map $H \rightarrow \text{Aut}(P)$ given by $h \mapsto \Phi_h$ is a group homomorphism.

This is essentially the definition adopted in [28]. Condition 3 is known to have undesirably restrictive consequences in some situations, such as when H is a translation group [29], but for our applications the definition will be satisfactory.

We take \mathcal{M} to be a static, spherically symmetric spacetime and G to be $\text{SU}(n)$. In this subsection we specify \mathcal{M} by a coordinate-based ansatz. The ansatz does not cover all regions of the Kruskal-type black and white whole spacetimes that will be introduced in Subsection 2.2, but we shall see that the ensuing gauge field will remain spherically symmetric when appropriately continued beyond the coordinate singularities.

The metric ansatz is

$$ds^2 = -Ne^{-2\delta}dt^2 + N^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

where the functions N and δ depend only on the coordinate r and we assume $N > 0$ and $r > 0$. The factor $d\theta^2 + \sin^2\theta d\phi^2$ is recognised as the metric on unit S^2 , where (θ, ϕ) are standard angle coordinates with coordinate singularities at $\theta = 0$ and $\theta = \pi$. It is evident from the ansatz that the metric has an $\text{SO}(3)$ isometry whose orbits are spacelike with topology S^2 and area $4\pi r^2$. We refer to this $\text{SO}(3)$ isometry as spherical symmetry.

The metric (2.1) is static, with the timelike Killing vector ∂_t that is orthogonal to the hypersurfaces of constant t . We refer to the coordinates (t, r, θ, ϕ) as Schwarzschild-like coordinates and to r as the area-radius. A systematic derivation of the ansatz (2.1) from the assumptions of spherical symmetry and staticity is given in [20]. The ansatz does not cover static, spherically symmetric spacetimes in which the area of the $\text{SO}(3)$ orbits is constant (see Section 2 in [20], Section 15.4 in [30] or Exercise 32.1 in [31]), but this special case does not occur within the black hole spacetimes in which we are interested.

The $\text{SO}(3)$ action on \mathcal{M} induces an action of the covering group $\text{SU}(2)$, by the double cover map $\text{SU}(2) \rightarrow \text{SU}(2)/\{\pm \text{Id}\} \simeq \text{SO}(3)$. Following [20, 21], we adopt $\text{SU}(2)$ as the group H of spherical symmetry in the above definition of a spherically symmetric connection. We shall now recall the resulting classification of these configurations and their description in an adapted Lie algebra basis [20].

The first part of the argument consists of determining all $\text{SU}(n)$ principal bundles that admit an $\text{SU}(2)$ action of the required kind. For spacetimes that are regularly foliated by the $\text{SO}(3)$ orbits, as is the case in (2.1), this amounts to classifying all $\text{SU}(n)$ principal bundles over S^2 . The classification relies on presenting S^2 as the quotient space $\simeq \text{SU}(2)/\text{U}(1)$ of the base space and analysing the action of the isotropy subgroup $\text{U}(1) \subset \text{SU}(2)$ on the total space of the bundle. The result is that, up to isomorphisms, the bundles are in one-to-one correspondence with the conjugacy classes of group homomorphisms from $\text{U}(1)$ to $\text{SU}(n)$ [28].

A convenient unique representative from each conjugacy class is the map $\lambda : \text{U}(1) \rightarrow \text{SU}(n)$, $z \mapsto \text{diag}(z^{k_1}, \dots, z^{k_n})$, where the n integers k_1, \dots, k_n satisfy $k_1 \geq k_2 \geq \dots \geq k_n$

and sum to zero. It follows that the equivalence classes of the $SU(n)$ principal bundles can be uniquely indexed by sets of n integers $\{k_1, \dots, k_n\}$ that sum to zero and are ordered so that $k_1 \geq k_2 \geq \dots \geq k_n$.

For the second part of the argument, one fixes the bundle and an $SU(2)$ action on it and considers all connections that are invariant under this action. Let the map $\lambda : U(1) \rightarrow SU(n)$ be as defined above, and let $\lambda' : \mathfrak{su}(2) \rightarrow \mathfrak{su}(n)$ denote the derivative of λ at the identity. A theorem of Wang [32] then states that the invariant connections are in one-to-one correspondence with the set of linear maps $\Lambda : \mathfrak{su}(2) \rightarrow \mathfrak{su}(n)$ satisfying the conditions

$$\Lambda(X) = \lambda'(X), \quad (2.2a)$$

$$\Lambda \circ \text{ad}_z = \text{ad}_{\lambda(z)} \circ \Lambda, \quad (2.2b)$$

for all $X \in \mathfrak{u}(1)$ and $z \in U(1)$, where $U(1)$ is again the isotropy subgroup of the $SU(2)$ action. The curvature F of these connections takes the form

$$F(\tilde{X}, \tilde{Y}) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]), \quad (2.3)$$

where $X, Y \in \mathfrak{su}(2)$ and \tilde{X}, \tilde{Y} are the corresponding vector fields induced by the $SU(2)$ action on the total space.

We adopt for $\mathfrak{su}(2)$ the basis $\tau_l := -\frac{i}{2}\sigma_l$, $l = 1, 2, 3$, where σ_l are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.4)$$

We write $\Lambda_l := \Lambda(\tau_l)$, $l = 1, 2, 3$, and we may without loss of generality choose the isotropy subgroup $U(1)$ to be embedded in $SU(2)$ as $z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. From (2.2a) it then follows that $\Lambda_3 = -\frac{i}{2} \text{diag}(k_1, \dots, k_n)$. The infinitesimal version of (2.2b) reads

$$\Lambda([\tau_3, \tau_l]) = [\Lambda_3, \Lambda_l], \quad i = 1, 2, \quad (2.5)$$

which implies that Λ_1 and Λ_2 can be written as

$$\Lambda_1 = \frac{1}{2} (C - C^H), \quad \Lambda_2 = -\frac{i}{2} (C + C^H), \quad (2.6)$$

where C is a strictly upper triangular complex $n \times n$ matrix, C^H is its hermitian conjugate, and $C_{ij} \neq 0$ if and only if $k_i = k_j + 2$.

Evaluating (2.3) on the $\mathfrak{su}(2)$ basis τ_l shows that the only non-vanishing component of the curvature form is $F(\tilde{\tau}_1, \tilde{\tau}_2) = [\Lambda_1, \Lambda_2] - \Lambda_3$. As the base space S^2 is two-dimensional, the curvature form must be proportional to the spherically symmetric volume form $\sin \theta d\theta \wedge d\phi$. The curvature form on S^2 must hence take the form

$$F = ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin \theta d\theta \wedge d\phi. \quad (2.7)$$

A corresponding connection form is

$$\hat{A} := \Lambda_1 d\theta + (\Lambda_2 \sin \theta + \Lambda_3 \cos \theta) d\phi. \quad (2.8)$$

Finally, the connection form A on the four-dimensional spacetime (2.1) can be decomposed as

$$A = \tilde{A} + \hat{A}, \quad (2.9)$$

where \hat{A} is as in (2.8) but the components of the matrix C in (2.6) are allowed to depend on the coordinates (t, r) . The remaining part \tilde{A} is an $\mathfrak{su}(n)$ -valued one-form on the two-dimensional spacetime obtained by dropping the angles from (2.1), invariant under the adjoint action of the subgroup $\lambda([\mathrm{U}(1)])$ [28].

In what follows we consider only the case [22, 23, 24] where the set of n integers is $\{k_1, \dots, k_n\} = \{n-1, n-3, n-5, \dots, -n+3, -n+1\}$. The connection form is taken to have a vanishing Coulomb component, $A_t = 0$, and one can then choose the gauge so that also the radial component A_r is zero. This means that we consider purely magnetic configurations of the form

$$A = \Lambda_1 d\theta + (\Lambda_2 \sin \theta + \Lambda_3 \cos \theta) d\phi, \quad (2.10)$$

where the traceless antihermitian matrices Λ_1 , Λ_2 and Λ_3 are given by

$$\Lambda_1 = \frac{1}{2} \begin{pmatrix} 0 & w_1 & & & & \\ -w_1 & 0 & w_2 & & & \\ & -w_2 & 0 & w_3 & & \\ & & \dots & \dots & \dots & \\ & & & -w_{n-2} & 0 & w_{n-1} \\ & & & & -w_{n-1} & 0 \end{pmatrix}, \quad (2.11a)$$

$$\Lambda_2 = -\frac{i}{2} \begin{pmatrix} 0 & w_1 & & & & \\ w_1 & 0 & w_2 & & & \\ & w_2 & 0 & w_3 & & \\ & & \dots & \dots & \dots & \\ & & & w_{n-2} & 0 & w_{n-1} \\ & & & & w_{n-1} & 0 \end{pmatrix}, \quad (2.11b)$$

$$\Lambda_3 = -\frac{i}{2} \begin{pmatrix} n-1 & & & & & \\ & n-3 & & & & \\ & & n-5 & & & \\ & & & \dots & & \\ & & & & -n+3 & \\ & & & & & -n+1 \end{pmatrix}, \quad (2.11c)$$

and the real-valued functions w_j , $j = 1, \dots, n-1$, depend only on the coordinate r .

We end the subsection with three comments.

First, the one-form (2.10) has a Dirac string singularity as $\theta \rightarrow 0$ and $\theta \rightarrow \pi$ [33]. The regularity of the curvature form (2.7) shows that this singularity is a gauge artefact. As the triviality of the fundamental group of $SU(n)$ implies that $SU(n)$ principal bundles over two-spheres are trivial [34, 35], the one-form (2.10) must therefore be a local representative of a connection one-form in the trivial $SU(n)$ bundle over the spacetime. We shall explicitly remove the Dirac string singularity in Subsection 3.3.

Second, the ansatz (2.10) has a residual gauge freedom in that a gauge transformation by

$$e^{ik\pi/n} \text{diag}(\underbrace{-1, -1, \dots, -1}_k, \underbrace{1, 1, \dots, 1}_{n-k}) \in SU(n) \quad (2.12)$$

leaves w_j invariant for $j \neq k$ but changes the sign of w_k [22]. We shall use this gauge freedom to simplify the special geon configurations that will be found in Subsection 3.2.

Third, we note that the embedding of the isotropy subgroup $U(1) \subset SU(2)$ in $SU(n)$ maps $\exp(2\pi\tau_3) = -\text{Id}_{SU(2)}$ to $\exp(2\pi\Lambda_3)$, which for our configurations (2.11) equals $\text{Id}_{SU(n)}$ for odd n and $-\text{Id}_{SU(n)}$ for even n . A gauge transformation by $\exp(2\pi\Lambda_3)$ hence leaves the ansatz (2.10) invariant, and by the discussion in Subsection 3.3 the same holds also in a globally regular gauge in which the Dirac string singularities of (2.10) have been removed. For the gauge configurations that we consider, the spherical symmetry action of $SU(2)$ hence projects to a spherical symmetry action of $SU(2)/\{\pm \text{Id}\} \simeq SO(3)$.

2.2 Nondegenerate Killing horizon: Kruskal-like extension

The metric (2.1) and the connection form (2.10) give an ansatz that can be inserted in the Einstein-Yang-Mills field equations. We are interested in spacetimes that have a nondegenerate Killing horizon at $r = r_h > 0$, where $N(r_h) = 0$ and $N'(r_h) > 0$, the prime indicating derivative with respect to r . Initial data for integrating the field equations from $r = r_h$ towards increasing r then consists of r_h , $\delta(r_h)$ and $w_j(r_h)$, $j = 1, \dots, n-1$. Local solutions in some neighbourhood of the horizon exist under a weak regularity restriction on $w_j(r_h)$ [22, 23]. Not all of these local solutions extend to an asymptotically flat (for a vanishing cosmological constant) or asymptotically anti-de Sitter (for a negative cosmological constant) infinity at $r \rightarrow \infty$, but for those that do, the solution is a static region of a nondegenerate black hole spacetime. Numerical results are given in [22, 23, 24, 36].

To extend the metric across the Killing horizon, we start in the exterior region and define the Kruskal-like coordinates (U, V, θ, ϕ) by

$$U := -\exp \left[-\alpha \left(t - \int_{r_0}^r \frac{e^{\delta(r)}}{N(r)} dr \right) \right], \quad (2.13a)$$

$$V := \exp \left[\alpha \left(t + \int_{r_0}^r \frac{e^{\delta(r)}}{N(r)} dr \right) \right], \quad (2.13b)$$

where $\alpha := \frac{1}{2}N'(r_h)e^{-\delta(r_h)}$ and the constant r_0 is chosen so that the product UV ,

$$UV = -\exp \left[2\alpha \int_{r_0}^r \frac{e^{\delta(r)}}{N(r)} dr \right], \quad (2.14)$$

has the Taylor expansion

$$UV = -\frac{r - r_h}{r_h} \left[1 + \left(\delta'(r_h) - \frac{1}{2} \frac{N''(r_h)}{N'(r_h)} \right) (r - r_h) + O((r - r_h)^2) \right] \quad (2.15)$$

as $r \rightarrow r_h$. It follows that in the exterior we have $U < 0$ and $V > 0$, and the Killing horizon is at $UV \rightarrow 0_-$. Whether UV is bounded below depends on the asymptotic behaviour of the metric at large r , but this will not affect what follows.

The metric in the coordinates (U, V, θ, ϕ) reads

$$ds^2 = \frac{1}{\alpha^2} \frac{N(r)e^{-2\delta(r)}}{UV} dUdV + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.16)$$

where r is a function of UV via (2.14). Inverting (2.15) as

$$\frac{r - r_h}{r_h} = -UV \left[1 + \left(\delta'(r_h) - \frac{1}{2} \frac{N''(r_h)}{N'(r_h)} \right) r_h UV + O((UV)^2) \right], \quad (2.17)$$

we find that the metric (2.16) has the near-horizon expansion

$$\begin{aligned} ds^2 = -\frac{4r_h}{N'(r_h)} & \left[1 + \left(3\delta'(r_h) - \frac{N''(r_h)}{N'(r_h)} \right) r_h UV + O((UV)^2) \right] dUdV \\ & + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (2.18)$$

which is regular across $UV = 0$. The metric can hence be extended from the original, ‘right-hand-side’ exterior to the black hole interior where $U > 0$ and $V > 0$, to the white hole interior where $U < 0$ and $V < 0$ and to the ‘left-hand-side’ exterior where $U > 0$ and $V < 0$. If the functions $N(r)$ and $\delta(r)$ are smooth at $r = r_h$, it further follows that the metric in the Kruskal coordinates is smooth at the horizon. Whether UV has an upper limit in the black and white hole regions, and whether there are further Killing horizons past these regions, will not affect what follows.

The extension of the gauge potential across the horizon is given by (2.10) and (2.11), with $w_j = w_j(r(UV))$. The extension is regular since $w_j(r_h)$ are part of the boundary data for the exterior solution, and the extension is smooth if $w_j(r)$ are smooth at $r = r_h$.

The resulting Kruskal-like extension is spherically symmetric, with the orbits of the $SO(3)$ isometry being spacelike and having topology S^2 . As the exterior gauge potential ansatz (2.10) does not have terms proportional to dt or dr , and as the coefficients depend only on r , the gauge field on the Kruskal-like extension is spherically symmetric in the same sense as in the exterior.

3 Geon quotient of the spherically symmetric $SU(n)$ black hole

We wish to take a geon quotient of the Kruskal-like $SU(n)$ black hole of Section 2. For the spacetime manifold this is a straightforward adaptation of the procedure with which the Kruskal manifold is quotiented into the \mathbb{RP}^3 geon [5, 6, 7, 8], and we shall review the requisite notions in Subsection 3.1. The new issues arise with including in the quotient the principal bundle in which the gauge field lives. These issues will be addressed in Subsections 3.2–3.5.

For presentational simplicity, we take the gauge group of the black hole bundle to be $SU(n)$ for odd n and $SU(n)/\{\pm \text{Id}\}$ for even n . We denote this gauge group by G . We write equations in G as matrix equations in the defining matrix representation, understanding for even n the matrices to be defined up to overall sign. Proceeding with the gauge group $SU(n)$ for all n would yield the same end results but we shall see in Subsection 3.3 that our choice of G will shorten the technical steps.

3.1 Geon quotient of the spacetime manifold

Let \mathcal{M} denote the spacetime manifold of the Kruskal-like extension, with the metric constructed in Subsection 2.2. \mathcal{M} is covered by the Kruskal-like coordinates (U, V, θ, ϕ) , with the usual coordinate singularities of the angle coordinates at $\theta = 0$ and $\theta = \pi$. In addition to the three Killing vectors of the $SO(3)$ isometry, \mathcal{M} has the Killing vector $\xi := V\partial_V - U\partial_U$, which is timelike in the right and left exterior quadrants where $UV < 0$, spacelike in the black and white hole quadrants where $UV > 0$ and null on the bifurcate Killing horizon $UV = 0$. In the right exterior quadrant where $V > 0$ and $U < 0$, covered by the metric (2.1), $\xi = \alpha^{-1}\partial_t$.

\mathcal{M} has topology $\mathbb{R}^2 \times S^2$ and is foliated by spacelike hypersurfaces of topology $\mathbb{R} \times S^2 \simeq S^3 \setminus \{\text{two points}\}$, each omitted point being at a spatial infinity. The product UV may be bounded below by some negative constant, depending on the nature of the spatial infinities [12], and it may be bounded above by some positive constant, depending on the properties of the spacetime in the black and white hole regions. The possible existence of such bounds will not affect what follows.

Consider now the map

$$\begin{aligned} J : \mathcal{M} &\rightarrow \mathcal{M}; \\ (U, V, \theta, \phi) &\mapsto (V, U, \pi - \theta, \phi + \pi), \end{aligned} \tag{3.1}$$

where the action on the angle coordinates is recognised as the S^2 antipodal map and is understood in this sense at the coordinate singularities. J is an involutive isometry without fixed points, and it preserves both space and time orientation. The quotient $\mathcal{M}' := \mathcal{M}/\{\text{Id}, J\}$ is therefore a time and space orientable spacetime. \mathcal{M}' is foliated by spacelike hypersurfaces of topology $(S^3 \setminus \{\text{two points}\})/\mathbb{Z}_2 \simeq \mathbb{RP}^3 \setminus \{\text{point}\}$, with the omitted point being at a spatial infinity. As recalled in Section 1, these properties

make \mathcal{M}' a topological geon spacetime, in the asymptotically flat case in the sense of Sorkin [1, 2, 3, 4] and in the asymptotically anti-de Sitter case in the generalised sense of [9, 10, 11, 12].

As the quotienting identifies the two exterior regions of \mathcal{M} , \mathcal{M}' is an eternal black and white hole spacetime, with a single exterior region that isometric to one exterior region of \mathcal{M} . We may hence refer to \mathcal{M}' as a topological geon black hole. The conformal diagram depends on the character of the spatial infinity and on the structure of the black hole interior: representative samples may be found in [8, 9, 14, 16].

We end with two observations on the isometries of \mathcal{M}' .

First, as the $SO(3)$ action on \mathcal{M} commutes with J , there is an induced $SO(3)$ action on \mathcal{M}' , with two-dimensional spacelike orbits. The generic orbits have again topology S^2 , but the special orbits that come from the $U = V$ subset of \mathcal{M} have topology \mathbb{RP}^2 . We shall regard \mathcal{M}' as a spherically symmetric spacetime despite these exceptional orbits.

Second, J changes the sign of the Killing vector ξ . The isometries generated by ξ on \mathcal{M} do therefore not induce an isometry on \mathcal{M}' : while such isometries exist within the exterior region of \mathcal{M}' , they cannot be extended past the horizon. The ramifications of this phenomenon for the Hawking-Unruh effect on related Einstein(-Maxwell) topological geon black holes have been investigated in [9, 10, 11, 14, 15, 18, 19].

3.2 Special gauge field configurations: geon quotient of the principal bundle

We now embark on the task of examining whether the spacetime quotient $\mathcal{M} \mapsto \mathcal{M}'$ can be extended to the principal bundle in which the gauge field lives.

Let A_{ext} denote the gauge potential (2.10) on \mathcal{M} ,

$$A_{\text{ext}} := \Lambda_1 d\theta + (\Lambda_2 \sin \theta + \Lambda_3 \cos \theta) d\phi. \quad (3.2)$$

We need to examine whether there is a bundle map that projects to J on \mathcal{M} and leaves the gauge field invariant. In terms of the gauge potential A_{ext} (3.2), this amounts to asking whether J maps A_{ext} to a gauge-equivalent gauge potential. Denoting by A_{ext}^J the pull-back of A_{ext} by J , we thus seek a gauge function $\Omega : \mathcal{M} \rightarrow G$, such that a gauge transformation by Ω maps A_{ext}^J back to A_{ext} ,

$$\Omega A_{\text{ext}}^J \Omega^{-1} + \Omega d\Omega^{-1} = A_{\text{ext}}. \quad (3.3)$$

From (3.1) we find

$$A_{\text{ext}}^J = -\Lambda_1 d\theta + (\Lambda_2 \sin \theta - \Lambda_3 \cos \theta) d\phi. \quad (3.4)$$

As neither (3.2) nor (3.4) involves dU or dV , we may assume Ω to depend only on the angular coordinates (θ, ϕ) . Equation (3.3) is then equivalent to the pair

$$-\Omega \Lambda_1 \Omega^{-1} + \Omega \partial_\theta \Omega^{-1} = \Lambda_1, \quad (3.5a)$$

$$\Omega (\Lambda_2 \sin \theta - \Lambda_3 \cos \theta) \Omega^{-1} + \Omega \partial_\phi \Omega^{-1} = \Lambda_2 \sin \theta + \Lambda_3 \cos \theta. \quad (3.5b)$$

To find a necessary condition for a solution to (3.5) to exist, we consider the field strengths of A_{ext} and A_{ext}^J . These can be computed from

$$F(X, Y) = dA(X, Y) + \frac{1}{2}[A(X), A(Y)], \quad (3.6)$$

with the result

$$\begin{aligned} F_{\text{ext}} &= \partial_U \Lambda_1 dU \wedge d\theta + \partial_V \Lambda_1 dV \wedge d\theta + \partial_U \Lambda_2 \sin \theta dU \wedge d\phi \\ &\quad + \partial_V \Lambda_2 \sin \theta dV \wedge d\phi + ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin \theta d\theta \wedge d\phi, \end{aligned} \quad (3.7a)$$

$$\begin{aligned} F_{\text{ext}}^J &= -\partial_U \Lambda_1 dU \wedge d\theta - \partial_V \Lambda_1 dV \wedge d\theta + \partial_U \Lambda_2 \sin \theta dU \wedge d\phi \\ &\quad + \partial_V \Lambda_2 \sin \theta dV \wedge d\phi - ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin \theta d\theta \wedge d\phi. \end{aligned} \quad (3.7b)$$

From (3.3) it follows that these field strengths are related by

$$\Omega F_{\text{ext}}^J \Omega^{-1} = F_{\text{ext}}. \quad (3.8)$$

Inserting (3.7) in (3.8) and using the fact that Ω only depends on the angular coordinates, (3.8) reduces to

$$\Omega \Lambda_1 \Omega^{-1} = -\Lambda_1, \quad (3.9a)$$

$$\Omega \Lambda_2 \Omega^{-1} = \Lambda_2, \quad (3.9b)$$

$$\Omega([\Lambda_1, \Lambda_2] - \Lambda_3) \Omega^{-1} = -([\Lambda_1, \Lambda_2] - \Lambda_3). \quad (3.9c)$$

Simplifying (3.9c) with the help of (3.9a) and (3.9b) shows that the set (3.9) is equivalent to

$$\Omega \Lambda_1 \Omega^{-1} = -\Lambda_1, \quad (3.10a)$$

$$\Omega \Lambda_2 \Omega^{-1} = \Lambda_2, \quad (3.10b)$$

$$\Omega \Lambda_3 \Omega^{-1} = -\Lambda_3. \quad (3.10c)$$

The set (3.10) is hence a necessary condition for (3.5) to hold.

To analyse (3.10), observe first from (2.11c) that Λ_3 and $-\Lambda_3$ are diagonal and their diagonal elements appear in the reverse order,

$$-\Lambda_3 = -\frac{i}{2} \begin{pmatrix} -n+1 & & & & & \\ & -n+3 & & & & \\ & & -n+5 & & & \\ & & & \ddots & & \\ & & & & n-3 & \\ & & & & & n-1 \end{pmatrix}. \quad (3.11)$$

Identity (3.10c) thus implies that Ω has the form

$$\Omega = (-i)^{n-1} \begin{pmatrix} & & & & \alpha_1 \\ & & & \alpha_2 & \\ & & \ddots & & \\ & \alpha_{n-1} & & & \\ \alpha_n & & & & \end{pmatrix} \quad (3.12)$$

where α_j are complex numbers with unit magnitude and $\prod_{j=1}^n \alpha_j = 1$.

Consider then (3.10a) and (3.10b). Using $\Omega^{-1} = \overline{\Omega}^T$, where the overline denotes complex conjugation and T transposition, we find

$$\Omega \Lambda_1 \Omega^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -\alpha_1 \overline{\alpha}_2 w_{n-1} & & & & \\ \alpha_2 \overline{\alpha}_1 w_{n-1} & 0 & -\alpha_2 \overline{\alpha}_3 w_{n-2} & & & \\ & \dots & \dots & \dots & & \\ & & \alpha_{n-1} \overline{\alpha}_{n-2} w_2 & 0 & -\alpha_{n-1} \overline{\alpha}_n w_1 & \\ & & & \alpha_n \overline{\alpha}_{n-1} w_1 & 0 & \end{pmatrix}. \quad (3.13)$$

By (2.11a) and (3.13), (3.10a) reduces to the set

$$\begin{aligned} \alpha_1 \overline{\alpha}_2 w_{n-1} &= w_1 = \alpha_2 \overline{\alpha}_1 w_{n-1}, \\ \alpha_2 \overline{\alpha}_3 w_{n-2} &= w_2 = \alpha_3 \overline{\alpha}_2 w_{n-2}, \\ &\vdots \\ \alpha_{n-2} \overline{\alpha}_{n-1} w_2 &= w_{n-2} = \alpha_{n-1} \overline{\alpha}_{n-2} w_2, \\ \alpha_{n-1} \overline{\alpha}_n w_1 &= w_{n-1} = \alpha_n \overline{\alpha}_{n-1} w_1, \end{aligned} \quad (3.14)$$

and it can be similarly verified that also (3.10b) reduces to (3.14).

As $n > 2$ by assumption, alphas satisfying (3.14) do not exist for generic gauge field configurations. There is however a special class of gauge field configurations for which such alphas exist. If w_j is vanishing, the j th line of (3.14) requires w_{n-j} to vanish. If w_j is nonvanishing, the j th line of (3.14) implies $w_{n-j} = \epsilon_j w_j$ and $\alpha_{j+1} = \epsilon_j \alpha_j$, where $\epsilon_j \in \{-1, +1\}$, and if n is even, $\epsilon_{n/2} = 1$. A necessary condition for the alphas to exist is therefore that the gauge potential functions satisfy $w_{n-j} = \epsilon_j w_j$ for all j , with $\epsilon_j \in \{-1, +1\}$ and $\epsilon_j = \epsilon_{n-j}$. Note that this condition is compatible with the radial evolution equation for the gauge potential functions [22]. As observed in Subsection 2.1, the sign of each w_j can be independently changed by a gauge transformation. The gauge can therefore be chosen so that the necessary condition for the alphas to exist reads

$$w_{n-j} = w_j, \quad \forall j. \quad (3.15)$$

Now, when (3.15) holds, (3.14) is solved by $\alpha_j = 1 \forall j$, and Ω (3.12) then takes the form

$$\Omega = (-i)^{n-1} \begin{pmatrix} & & & 1 \\ & & \dots & 1 \\ & 1 & & \\ 1 & & & \end{pmatrix}. \quad (3.16)$$

The condition (3.15) is hence also sufficient for a solution to (3.10) to exist, and as (3.16) is independent of the angles, this also provides a solution to (3.5).

We summarise. The necessary and sufficient condition for a geon quotient with the gauge group G to exist is (3.15), up to gauge transformations. When (3.15) holds, the gauge transformation that compensates for J in the quotienting bundle map is given by (3.16).

We note in passing that for $n = 2$ the only gauge potential function is w_1 and the equations (3.14) have the solution $\alpha_1 = \alpha_2 = 1$. This yields the purely magnetic special case of the $SU(2)$ geon described in [12].

3.3 Triviality of the black hole bundle

Up to now we have been working in a gauge in which the gauge potential A_{ext} (3.2) has Dirac string singularities at $\theta = 0$ and $\theta = \pi$. As noted at the end of Subsection 2.1, the gauge bundle over the Kruskal-like spacetime \mathcal{M} is trivial, and a globally regular gauge on \mathcal{M} must hence exist. In this subsection we transform A_{ext} into a globally regular gauge. This will be used in Subsection 3.4 to analyse the gauge bundle over the geon spacetime.

To begin, observe that gauge transformations by the functions

$$\Omega_N := \text{diag}(e^{-i(n-1)\phi/2}, e^{-i(n-3)\phi/2}, \dots, e^{-i(-n+3)\phi/2}, e^{-i(-n+1)\phi/2}), \quad (3.17a)$$

$$\Omega_S := \text{diag}(e^{i(n-1)\phi/2}, e^{i(n-3)\phi/2}, \dots, e^{i(-n+3)\phi/2}, e^{i(-n+1)\phi/2}), \quad (3.17b)$$

make the gauge potential A_{ext} (3.2) regular everywhere except respectively at $\theta = \pi$ and $\theta = 0$. This is the step where taking the gauge group to be $SU(n)/\{\pm \text{Id}\}$ for even n shortens the discussion, as the expressions (3.17) are not single-valued in $SU(n)$ for even n .

It therefore suffices to find a gauge function $H : S^2 \setminus (\{\theta = 0\} \cup \{\theta = \pi\})$ that agrees with Ω_N in some punctured neighbourhood of $\theta = 0$, agrees with Ω_S in some punctured neighbourhood of $\theta = \pi$, and interpolates in between: a transformation by H puts A_{ext} (3.2) into a globally regular gauge. We shall show that such gauge functions exist.

Let first n be odd. The formulas (3.17) for Ω_N and Ω_S define two paths in $G = SU(n)$, with path parameter $\phi \in [0, 2\pi]$. These paths are closed, starting and ending at the identity. As the fundamental group of $SU(n)$ is trivial [37], these paths are homotopic, and any homotopy between them, with θ as the homotopy parameter (for example with $\pi/2 \leq \theta \leq 3\pi/4$), provides the interpolation we need.

Let then n be even. The formulas (3.17) for Ω_N and Ω_S again define two closed paths in $G = SU(n)/\{\pm \text{Id}\}$, starting and ending at the identity, with path parameter $\phi \in [0, 2\pi]$. When these paths are lifted from G to its double cover $SU(n)$, formulas (3.17) show that each lift starts at $\text{Id} \in SU(n)$ and ends at $-\text{Id} \in SU(n)$. As the fundamental group of $SU(n)$ is trivial, these two lifts are homotopic to each other in $SU(n)$, and this homotopy in $SU(n)$ projects down into a homotopy between the original closed paths in $G = SU(n)/\{\pm \text{Id}\}$. Hence the homotopy between the closed paths in G provides again the interpolation we need.

Finally, note that for even n a connection in the trivial $SU(n)/\{\pm \text{Id}\}$ bundle lifts into a connection in the trivial $SU(n)$ bundle. Using the gauge group $SU(n)/\{\pm \text{Id}\}$ instead of $SU(n)$ for even n is hence just a presentational convenience.

3.4 Triviality of the geon bundle for the configurations of Subsection 3.2

In this subsection we show that the geons of Subsection 3.2 have a trivial gauge bundle.

We showed in Subsection 3.3 that the black hole bundle P is trivial and we can realise it as $P := \mathcal{M} \times G$. In this realisation, the geon bundle P' is the quotient of P by the \mathbb{Z}_2 group of bundle automorphisms whose nontrivial element K takes the form

$$\begin{aligned} K : \mathcal{M} \times G &\rightarrow \mathcal{M} \times G; \\ (x, h) &\mapsto (J(x), h \cdot \Xi(x)^{-1}), \end{aligned} \tag{3.18}$$

where $\Xi : \mathcal{M} \rightarrow G$ is the gauge function that compensates for J in a globally regular gauge. The G -multiplication denoted by a dot is matrix multiplication for odd n and matrix multiplication up to overall sign for even n .

We shall work in the globally regular gauge that is obtained from the gauge (3.15) by the procedure of Subsection 3.3. In this gauge we have

$$\Xi(x) = H(x)\Omega[H(J(x))]^{-1} \tag{3.19}$$

for $0 < \theta < \pi$, where Ω is given by (3.16) and H was defined in Subsection 3.3. It follows from (3.16) and (3.17) that Ξ takes a constant value in sufficiently small punctured neighbourhoods of $\theta = 0$ and $\theta = \pi$. Ξ is therefore well defined on \mathcal{M} , by (3.19) for $0 < \theta < \pi$ and by continuity at $\theta = 0$ and $\theta = \pi$.

Recall that a principal bundle is trivial iff it admits a global section. The geon bundle P' admits a global section iff P admits a global section σ that is invariant under K . By (3.18), this invariance condition reads

$$\sigma(J(x)) = \sigma(x) \cdot \Xi(x)^{-1}, \quad \forall x \in \mathcal{M}. \tag{3.20}$$

As the gauge potential depends on U and V only through the combination UV , it suffices to consider the condition (3.20) on the two-sphere at $U = V = 0$. It further suffices to consider (3.20) on the equator $\theta = \pi/2$ of the two-sphere. To see this, let γ and Ξ_{eq} denote the respective restrictions of σ and Ξ to the equator. The restriction of (3.20) to the equator then reads

$$\gamma(\phi + \pi) = \gamma(\phi) \cdot \Xi_{\text{eq}}(\phi)^{-1}. \tag{3.21}$$

If σ exists, it defines a solution to (3.21) by restriction. Conversely, suppose that a solution to (3.21) exists. We can view γ equivalently as a G -valued function on S^1 or as a closed path in G , denoted by the same letter and given by $\gamma : [0, 2\pi] \rightarrow G$;

$\phi \mapsto \gamma(\phi)$. When viewed as a closed path, γ is contractible. For odd n this follows because $G = \mathrm{SU}(n)$ has a trivial fundamental group. For even n the fundamental group of $G = \mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ is \mathbb{Z}_2 , but γ is contractible by the observation made in the last paragraph of Subsection 3.3, or alternatively by the explicit construction of γ below. Given γ , we can define σ for $0 \leq \theta \leq \pi/2$ by an arbitrary contraction of γ into a trivial path at $\theta = 0$. Defining σ for $\pi/2 < \theta \leq \pi$ by (3.20) then gives the desired σ .

What hence remains is to show that a solution to (3.21) exists. We now proceed to construct such a solution.

Let H_{eq} denote the restriction of H to the equator. From (3.19) we have $\Xi_{\mathrm{eq}}(\phi) = H_{\mathrm{eq}}(\phi)\Omega[H_{\mathrm{eq}}(\phi + \pi)]^{-1}$. Defining

$$\tilde{\gamma}(\phi) := \gamma(\phi) \cdot H_{\mathrm{eq}}(\phi + \pi), \quad (3.22)$$

the condition (3.21) can be rearranged into

$$\tilde{\gamma}(\phi + \pi) = \tilde{\gamma}(\phi) \cdot \Omega^{-1}. \quad (3.23)$$

Without loss of generality, we may set $\tilde{\gamma}(0) = \mathrm{Id}$; then $\tilde{\gamma}(\pi) = \Omega^{-1}$.

Since Ω^{-1} is special unitary, it can be diagonalised by

$$\Omega^{-1} = UDU^{-1}, \quad (3.24)$$

where U is unitary and D is a diagonal special unitary matrix whose diagonal elements are the eigenvalues of Ω^{-1} . We need to analyse these eigenvalues.

Let n be odd. A recursive evaluation of the determinant shows that $|\Omega^{-1} - \lambda \mathrm{Id}| = -(\lambda^2 - 1)^{(n-1)/2}(\lambda - (-1)^{(n-1)/2})$. The eigenvalues of Ω^{-1} are hence ± 1 , and the multiplicity of -1 is even. We now define the ϕ -dependent matrix $\hat{D}(\phi)$ by replacing an arbitrarily-chosen half of the -1 s in D by $e^{i\phi}$ and the other half by $e^{-i\phi}$. It is immediate that $\hat{D}(\phi) \in G$, \hat{D} has period 2π , $\hat{D}(0) = \mathrm{Id}$ and $\hat{D}(\pi) = D$. Given \hat{D} , we define $\tilde{\gamma}(\phi) := U\hat{D}(\phi)U^{-1}$. Then $\tilde{\gamma}(\phi) \cdot \Omega^{-1} = U\hat{D}(\phi)U^{-1}UDU^{-1} = U\hat{D}(\phi)DU^{-1} = U\hat{D}(\phi)\hat{D}(\pi)U^{-1} = U\hat{D}(\phi + \pi)U^{-1} = \tilde{\gamma}(\phi + \pi)$, so that $\tilde{\gamma}$ satisfies (3.23) and γ satisfies (3.21).

Let then n be even. Proceeding as above, we find $|\Omega^{-1} - \lambda \mathrm{Id}| = (\lambda^2 + 1)^{n/2}$. The eigenvalues of Ω^{-1} are hence $\pm i$, each with multiplicity $n/2$. We now define $\hat{D}(\phi)$ by replacing in D the eigenvalues i by $e^{i\phi/2}$ and the eigenvalues $-i$ by $e^{-i\phi/2}$. Then $\hat{D}(0) = \mathrm{Id}$, $\hat{D}(\pi) = D$, and although \hat{D} is not 2π -periodic as an $\mathrm{SU}(n)$ matrix, it is as a $G = \mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ matrix. Defining again $\tilde{\gamma}(\phi) := U\hat{D}(\phi)U^{-1}$, the conditions (3.23) and (3.21) can be verified as for odd n .

Finally, for even n , we verify explicitly the claim that the path $\gamma : [0, 2\pi] \rightarrow G$; $\phi \mapsto \gamma(\phi)$ constructed above is contractible in G . Without loss of generality, the gauge function H can be chosen to equal Ω_N (3.17a) on the equator. In this gauge it is transparent that the lift of H_{eq} into $\mathrm{SU}(n)$ is periodic in ϕ with period 4π and changes sign after a translation in ϕ by 2π . From (3.22) it follows that the lift of γ to $\mathrm{SU}(n)$ is a closed path in $\mathrm{SU}(n)$, and the contraction of this lift in $\mathrm{SU}(n)$ projects down to a contraction of γ in G .

This completes the proof of triviality of the geon bundle.

3.5 Generic gauge field configurations: geon quotient with gauged charge conjugation

We saw in Subsection 3.2 that a geon quotient with gauge group G does not exist for generic gauge field configurations. A similar obstacle for the Maxwell gauge field in the Reissner-Nordström black hole [5] can be overcome by promoting $U(1)$ charge conjugation from a global symmetry into a gauge symmetry [12]. In this subsection we show that a similar gauging of charge conjugation works also for the $SU(n)$ black holes at hand.

In the abelian case, the usual Maxwell gauge group $U(1) \simeq SO(2)$ is enlarged into $O(2) \simeq \mathbb{Z}_2 \ltimes SO(2) \simeq \mathbb{Z}_2 \ltimes U(1)$. In the $\mathbb{Z}_2 \ltimes U(1)$ representation, the group multiplication law reads

$$(a_1, u_1) \cdot (a_2, u_2) = (a_1 a_2, u_1 \rho_{a_1}(u_2)), \quad (3.25)$$

where $a_i \in \mathbb{Z}_2$, $u_i \in U(1)$, and $\rho : \mathbb{Z}_2 \rightarrow \text{Aut}(U(1))$, $a \mapsto \rho_a$, is the group homomorphism for which the nontrivial element of \mathbb{Z}_2 acts on $U(1)$ by complex conjugation. Writing $\mathbb{Z}_2 \simeq \{0, 1\}$, where the identity element is 0, the explicit formula for ρ is

$$\rho_0(u) = u, \quad (3.26a)$$

$$\rho_1(u) = \overline{u}. \quad (3.26b)$$

In the nonabelian case at hand, the original gauge group G is $SU(n)$ for odd n and $SU(n)/\{\pm \text{Id}\}$ for even n . We enlarge G to $G_{\text{enl}} := \mathbb{Z}_2 \ltimes G$ by (3.25) and (3.26). The group multiplication table of G_{enl} reads

$$\begin{aligned} (0, u_1) \cdot (0, u_2) &= (0, u_1 u_2), \\ (0, u_1) \cdot (1, u_2) &= (1, u_1 u_2), \\ (1, u_1) \cdot (0, u_2) &= (1, u_1 \overline{u}_2), \\ (1, u_1) \cdot (1, u_2) &= (0, u_1 \overline{u}_2). \end{aligned} \quad (3.27)$$

If Ω is a gauge function with values in G , it follows that the gauge function $\tilde{\Omega} := (a, \Omega) : \mathcal{M} \rightarrow G_{\text{enl}}$ transforms the gauge potential by

$$A \mapsto \tilde{\Omega} A \tilde{\Omega}^{-1} + \tilde{\Omega} d \tilde{\Omega}^{-1} = \begin{cases} \Omega A \Omega^{-1} + \Omega d \Omega^{-1} & \text{if } \tilde{\Omega} = (0, \Omega), \\ \Omega \bar{A} \Omega^{-1} + \Omega d \Omega^{-1} & \text{if } \tilde{\Omega} = (1, \Omega). \end{cases} \quad (3.28)$$

To find a geon, we follow Subsection 3.2 with Ω replaced by $\tilde{\Omega}$. The conditions (3.10) are replaced by

$$\tilde{\Omega} \Lambda_1 \tilde{\Omega}^{-1} = -\Lambda_1, \quad (3.29a)$$

$$\tilde{\Omega} \Lambda_2 \tilde{\Omega}^{-1} = \Lambda_2, \quad (3.29b)$$

$$\tilde{\Omega} \Lambda_3 \tilde{\Omega}^{-1} = -\Lambda_3. \quad (3.29c)$$

It follows from (2.11) and (3.28) that the set (3.29) is solved by $\tilde{\Omega} = (1, \Omega)$, where

$$\begin{aligned}\Omega &= \text{diag}(i^{-n+1}, i^{-n+3}, \dots, i^{n-3}, i^{n-1}) \\ &= (-i)^{n-1} \text{diag}(1, -1, 1, -1, \dots, (-1)^{n-1}).\end{aligned}\quad (3.30)$$

Hence the black hole bundle now admits a geon quotient without restrictions on the gauge field configuration.

If desired, the geon quotient can be described as in Subsection 3.4, by adopting in the trivial black hole bundle $\mathcal{M} \times G_{\text{enl}}$ a globally regular gauge. Now, however, the geon bundle is not trivial, since the gauge transformation part of the bundle map is in the disconnected component of G_{enl} .

4 Axially symmetric $\text{SU}(2)$ black holes

In this section we first review the static, axially symmetric Einstein- $\text{SU}(2)$ black holes discovered by Kleinhans and Kunz [25, 26]. For a generalisation to a negative cosmological constant, see [38]. We then give Kruskal-like coordinates that extend the spacetime across the horizon.

4.1 The exterior solution of Kleinhans and Kunz

A static, axially symmetric metric can be written in the isotropic coordinates (t, r, θ, ϕ) as

$$ds^2 = -f dt^2 + \frac{m}{f} dr^2 + \frac{m}{f} r^2 d\theta^2 + \frac{l}{f} r^2 \sin^2 \theta d\phi^2, \quad (4.1)$$

where the positive functions f , m and l depend only on r and θ . Here θ and ϕ are the usual angular coordinates on the (topological) S^2 , with coordinate singularities at $\theta = 0$ and $\theta = \pi$; for regularity of the spacetime at these coordinate singularities, we need $l/m \rightarrow 1$ as $\theta \rightarrow 0$ and as $\theta \rightarrow \pi$. The spacetime is static, with the timelike hypersurface-orthogonal Killing vector ∂_t . The Killing vector of axial symmetry is ∂_ϕ , with the symmetry axis at $\theta = 0$ and $\theta = \pi$.

The ansatz for the gauge potential is

$$A = \frac{1}{2er} \left[\tau_\phi^n (H_1 dr + (1 - H_2) r d\theta) - n (\tau_r^n H_3 + \tau_\theta^n (1 - H_4)) r \sin \theta d\phi \right], \quad (4.2)$$

where e is the coupling constant, the functions H_i depend only on r and θ , and

$$\tau_r^n := \sin \theta \cos n\phi \tau^x + \sin \theta \sin n\phi \tau^y + \cos \theta \tau^z, \quad (4.3a)$$

$$\tau_\theta^n := \cos \theta \cos n\phi \tau^x + \cos \theta \sin n\phi \tau^y - \sin \theta \tau^z, \quad (4.3b)$$

$$\tau_\phi^n := -\sin n\phi \tau^x + \cos n\phi \tau^y, \quad (4.3c)$$

where n is a positive integer and, to conform to the notation of [25, 26], τ^x , τ^y and τ^z denote respectively the Pauli matrices σ_1 , σ_2 and σ_3 (2.4). This ansatz is purely magnetic, with no term proportional to dt . The ansatz is static, containing no dependence

on t , and it is axially symmetric, in the sense that the rotation $\phi \mapsto \phi + \alpha$ can be undone by a gauge transformation with $\exp[-in(\alpha/2)\tau^z]$. With a 2π rotation in ϕ , the ansatz undergoes a $2\pi n$ rotation in $\mathfrak{su}(2)$: we hence refer to n as the *winding number*.

Finally, we require both the metric and the gauge field to be invariant, in an appropriate sense, under the north-south reflection $\theta \mapsto \pi - \theta$. For the metric the sense is that of isometry, implying that f , m and l are even under $\theta \mapsto \pi - \theta$. For the gauge field the sense is [26] that H_1 and H_3 are odd and H_2 and H_4 are even under $\theta \mapsto \pi - \theta$.

We are interested in solutions to the Einstein-SU(2) field equations with a nondegenerate Killing horizon of the Killing vector ∂_t at $r = r_h > 0$. The boundary conditions at the horizon and at the symmetry axis and the integration of the field equations into the exterior region $r > r_h$ were discussed in [25, 26, 39, 40, 41], and numerical evidence was found that solutions exist, including solutions that have an asymptotically flat infinity at $r \rightarrow \infty$. The defining properties of the nondegenerate horizon in the isotropic coordinates of the anzatz (4.1) are $f(r_h, \theta) = 0 = f'(r_h, \theta)$ and $f''(r_h, \theta) > 0$, where the prime indicates derivative with respect to r . Working in the dimensionless variable $\delta := (r/r_h - 1)$, it follows that the near-horizon Taylor expansions of the metric functions and the gauge potential functions begin

$$f(\delta, \theta) = f_2(\theta)\delta^2 \left[1 - \delta + \frac{1}{24}\delta^2 F(\theta) + O(\delta^3) \right], \quad (4.4a)$$

$$m(\delta, \theta) = m_2(\theta)\delta^2 \left[1 - 3\delta + \frac{1}{24}\delta^2 M(\theta) + O(\delta^3) \right], \quad (4.4b)$$

$$l(\delta, \theta) = l_2(\theta)\delta^2 \left[1 - 3\delta + \frac{1}{12}\delta^2 L(\theta) + O(\delta^3) \right], \quad (4.4c)$$

$$H_1(\delta, \theta) = \delta \left(1 - \frac{\delta}{2} \right) H_{11}(\theta) + O(\delta^3), \quad (4.5a)$$

$$H_2(\delta, \theta) = H_{20}(\theta) + \frac{1}{4}\delta^2 H_{21}(\theta) + O(\delta^3), \quad (4.5b)$$

$$H_3(\delta, \theta) = H_{30}(\theta) + \frac{1}{8}\delta^2 H_{31}(\theta) + O(\delta^3), \quad (4.5c)$$

$$H_4(\delta, \theta) = H_{40}(\theta) + \frac{1}{8}\delta^2 H_{41}(\theta) + O(\delta^3), \quad (4.5d)$$

where the O -terms may depend on θ and the field equations yield various relations among the coefficient functions [26]. One of these relations is

$$\frac{1}{m_2} \frac{dm_2}{d\theta} - \frac{2}{f_2} \frac{df_2}{d\theta} = 0, \quad (4.6)$$

from which it follows that f_2^2/m_2 is independent of θ , implying that the horizon has constant surface gravity [26]. The gauge potential can further be chosen regular everywhere, including $\theta = 0$ and $\theta = \pi$ [40, 41]. The SU(2) bundle is thus trivial and the gauge potential is expressed in a globally regular gauge.

In the special case $n = 1$ the field equations imply that $l = m$, $H_1 = H_3 = 0$, $H_2 = H_4$ and all the metric and gauge potential functions are independent of θ . The metric and the gauge field are then spherically symmetric, and the solution reduces to that of [42, 43].

4.2 Kruskal-like extension

A complication with finding Kruskal-like coordinates that cover a neighbourhood of the full bifurcate Killing horizon is that the null geodesics with constant ϕ generically have nontrivial evolution in both r and θ . However, because of the discrete isometry $\theta \mapsto \pi - \theta$, the submanifold at $\theta = \pi/2$ is totally geodesic, and Kruskal-like coordinates that extend this submanifold across the horizon can be found as in the spherically symmetric case of Subsection 2.2. We shall show that the Kruskal-like coordinates adapted to the $\theta = \pi/2$ submanifold can be extended to other values of θ to give a C^0 extension across the horizon. This C^0 extension will suffice for taking the geon quotient in Section 5.

We start at $r > r_h$ and define the coordinates (U, V, θ, ϕ) by

$$U := -\exp \left[-\alpha \left(t - \int_{r_0}^r \frac{\sqrt{m(r, \pi/2)}}{f(r, \pi/2)} dr \right) \right], \quad (4.7a)$$

$$V := \exp \left[\alpha \left(t + \int_{r_0}^r \frac{\sqrt{m(r, \pi/2)}}{f(r, \pi/2)} dr \right) \right], \quad (4.7b)$$

where

$$\alpha := \frac{f_2(\pi/2)}{r_h \sqrt{m_2(\pi/2)}} \quad (4.8)$$

and r_0 is chosen so that

$$\int_{r_0}^r \frac{\sqrt{m(r, \pi/2)}}{f(r, \pi/2)} dr = \frac{1}{\alpha} \left[\ln \delta - \frac{1}{2} \delta + O(\delta^2) \right] \quad (4.9)$$

as $r \rightarrow r_h$. The region $r > r_h$ is at $U < 0$ and $V > 0$, and the Killing horizon is at $UV \rightarrow 0_-$. The metric in the coordinates (U, V, θ, ϕ) reads

$$\begin{aligned} ds^2 = & f(r, \theta) \frac{1}{2\alpha^2} \frac{1}{UV} \left[\frac{m(r, \theta)}{f(r, \theta)^2} \frac{f(r, \pi/2)^2}{m(r, \pi/2)} + 1 \right] dU dV \\ & + f(r, \theta) \frac{1}{4\alpha^2} \frac{1}{(UV)^2} \left[\frac{m(r, \theta)}{f(r, \theta)^2} \frac{f(r, \pi/2)^2}{m(r, \pi/2)} - 1 \right] (V^2 dU^2 + U^2 dV^2) \\ & + \frac{m(r, \theta)}{f(r, \theta)} r^2 d\theta^2 + \frac{l(r, \theta)}{f(r, \theta)} r^2 \sin^2 \theta d\phi^2, \end{aligned} \quad (4.10)$$

where r is a function of UV by

$$UV = -\exp \left[2\alpha \int_{r_0}^r \frac{\sqrt{m(r, \pi/2)}}{f(r, \pi/2)} dr \right] \quad (4.11a)$$

$$= -\delta^2 [1 - \delta + O(\delta^2)], \quad r \rightarrow r_h. \quad (4.11b)$$

Inverting the near-horizon expansion (4.11b) and substituting in (4.10) yields

$$\begin{aligned} ds^2 = & -\frac{1}{\alpha^2} f_2(\theta) [1 + O(UV)] dU dV + \frac{1}{96\alpha^2} f_2(\theta) \times \\ & \times \left[M(\theta) - M(\pi/2) - 2F(\theta) + 2F(\pi/2) + O(\sqrt{-UV}) \right] (V^2 dU^2 + U^2 dV^2) \\ & + \frac{1}{\alpha^2} f_2(\theta) [1 + O(UV)] d\theta^2 + \frac{l_2(\theta)}{f_2(\theta)} [1 + O(UV)] \sin^2 \theta d\phi^2. \end{aligned} \quad (4.12)$$

Similarly, the near-horizon expansion of the gauge potential (4.2) reads

$$\begin{aligned} A = & \frac{1}{2e} \left\{ \tau_\phi^n \left[-\frac{1}{2} (1 + O(UV)) H_{11}(\theta) (V dU + U dV) \right. \right. \\ & \left. \left. + (1 - H_{20}(\theta) + O(UV)) d\theta \right] \right. \\ & \left. - n \left[\tau_r^n (H_{30}(\theta) + O(UV)) + \tau_\theta^n (1 - H_{40} + O(UV)) \right] \sin \theta d\phi \right\}. \end{aligned} \quad (4.13)$$

The components of the metric (4.12) and the gauge potential (4.13) are well defined at the horizon, $UV \rightarrow 0_-$, but the components of the metric are not guaranteed to be differentiable because of the $O(\sqrt{-UV})$ error term. Our Kruskal coordinates therefore give a C^0 extension of the spacetime into a neighbourhood of the bifurcate Killing horizon, but they are not sufficiently regular for discussing the field equations across the horizon. Coordinates that allow a smooth extension are discussed in [44, 45], but at the expense of rendering the discrete isometry that we wish to utilise less transparent. We shall work with the above C^0 extension.

5 Geon quotient of the axially symmetric $SU(2)$ black hole

In this section we show that the Kruskal-like $SU(2)$ black hole of Section 4 has a geon quotient. As in Section 3, quotienting the spacetime manifold proceeds as taking the \mathbb{RP}^3 geon quotient of Kruskal [5, 6, 7, 8] and the issues of interest to us arise with quotienting the principal bundle in which the gauge field lives.

5.1 Spacetime quotient

Let \mathcal{M} denote the spacetime manifold of the Kruskal-like (C^0) extension covered by the coordinates (U, V, θ, ϕ) , with the coordinate singularities at $\theta = 0$ and $\theta = \pi$ understood to be handled as in Section 3. The map J defined by (3.1) is an involutive isometry without fixed points, preserving both space and time orientation and mapping the two exterior regions of \mathcal{M} to each other. The quotient spacetime $\mathcal{M}' := \mathcal{M}/\{\text{Id}, J\}$ is hence a time and space orientable black and white hole spacetime, its single exterior region is isometric to one exterior region of \mathcal{M} , and it is foliated by spacelike hypersurfaces of topology $\mathbb{RP}^3 \setminus \{\text{point}\}$ with the omitted point being at an asymptotically flat spatial infinity. \mathcal{M}' is hence a topological geon in the sense of Sorkin [1, 2, 3, 4] and we may refer to it as a topological geon black hole.

The isometries of \mathcal{M}' may be discussed as in Subsection 3.1. In particular, the Killing vector ∂_ϕ of \mathcal{M} is invariant under J and there is hence an induced $U(1)$ isometry group on \mathcal{M}' , with subtleties at the orbits coming from the subset of \mathcal{M} where $U = V$ and $\theta = \pi/2$. We shall regard \mathcal{M}' as an axially symmetric spacetime despite these exceptional orbits. The isometry properties associated with the Killing vector $V\partial_V - U\partial_U$ of \mathcal{M} are as in Subsection 3.1.

5.2 Principal bundle quotient

Let A_{ext} denote the gauge potential on \mathcal{M} , given in the right-hand-side exterior by (4.2) and having the near-horizon form (4.13). We need to investigate whether there exists a bundle map that projects to J on \mathcal{M} and leaves the gauge potential invariant. As in Section 3, this reduces to examining whether A_{ext} is invariant under J up to a gauge transformation.

From the evenness of the gauge potential functions H_2 and H_4 and the oddness of the gauge potential functions H_1 and H_3 under $\theta \mapsto \pi - \theta$, and from the properties of the matrices (4.3) under J , it follows that the cases of odd and even n require separate treatment.

Let first n be odd. A_{ext} is then clearly invariant under J , and the bundle map can be chosen to be

$$\begin{aligned} K_{\text{odd}} : \mathcal{M} \times \text{SU}(2) &\rightarrow \mathcal{M} \times \text{SU}(2); \\ (U, V, \theta, \phi, h) &\mapsto (V, U, \pi - \theta, \phi + \pi, h). \end{aligned} \tag{5.1}$$

K_{odd} is involutive, and the quotient bundle is the trivial $\text{SU}(2)$ bundle over $\mathcal{M}_g := \mathcal{M}/\{\text{Id}, J\}$. As the gauge potential is invariant under a gauge transformation by $-\text{Id} \in \text{SU}(2)$, the geon bundle can be alternatively taken to be the trivial $\text{SO}(3) \simeq \text{SU}(2)/\{\pm \text{Id}\}$ bundle over \mathcal{M}_g .

Let then n be even, and let A_{ext}^J denote the pull-back of A_{ext} by J . In the right-

hand-side exterior covered by the coordinates (t, r, θ, ϕ) , A_{ext}^J takes the form

$$\begin{aligned} A_{\text{ext}}^J = \frac{1}{2er} & \left\{ (\tau^x \sin n\phi - \tau^y \cos n\phi) [H_1 dr + (1 - H_2) r d\theta] \right. \\ & - n [(-\tau^x \sin \theta \cos n\phi - \tau^y \sin \theta \sin n\phi + \tau^z \cos \theta) H_3 \right. \\ & \left. + (-\tau^x \cos \theta \cos n\phi - \tau^y \cos \theta \sin n\phi - \tau^z \sin \theta) (1 - H_4)] r \sin \theta d\phi \right\}. \end{aligned} \quad (5.2)$$

Comparison with (4.2) shows that A_{ext}^J and A_{ext} do not coincide. They are however taken to each other by $(\tau^x, \tau^y, \tau^z) \mapsto (-\tau^x, -\tau^y, \tau^z)$, which is a gauge transformation: defining

$$g_0 := \exp \left(i \frac{\pi}{2} \tau^z \right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \text{SU}(2), \quad (5.3)$$

we have

$$A_{\text{ext}} = g_0 A_{\text{ext}}^J g_0^{-1}, \quad (5.4)$$

and (5.4) is a gauge transformation because the inhomogeneous term involving dg_0 vanishes. The bundle map can thus be chosen to be

$$\begin{aligned} K_{\text{ev}} : \mathcal{M} \times \text{SU}(2) & \rightarrow \mathcal{M} \times \text{SU}(2); \\ (U, V, \theta, \phi, h) & \mapsto (V, U, \pi - \theta, \phi + \pi, h \cdot g_0^{-1}), \end{aligned} \quad (5.5)$$

where the dot denotes matrix multiplication in $\text{SU}(2)$. K_{ev} generates the cyclic group of order four, $\bar{\Gamma} := \{\text{Id}, K_{\text{ev}}, K_{\text{ev}}^2, K_{\text{ev}}^3\}$, and the geon bundle is the quotient $(\mathcal{M} \times \text{SU}(2)) / \bar{\Gamma}$. As the normal subgroup $\{\text{Id}, K_{\text{ev}}^2\} \subset \bar{\Gamma}$ identifies points in $\mathcal{M} \times \text{SU}(2)$ by the position-independent gauge transformation by $g_0^2 = -\text{Id} \in \text{SU}(2)$, and as this gauge transformation leaves the gauge potential invariant, the geon bundle can be equivalently presented as a \mathbb{Z}_2 quotient of the trivial $\text{SO}(3) \simeq \text{SU}(2) / \{\pm \text{Id}\}$ bundle over \mathcal{M} . Explicitly, we may realise the projection $\text{SU}(2) \rightarrow \text{SO}(3)$, $g \mapsto \hat{g}$, in the defining matrix representations so that $g \tau^i g^{-1} = \sum_j \hat{g}^i_j \tau^j$. Note that $\hat{g}_0 = \text{diag}(-1, -1, 1)$. The involutive bundle map then reads

$$\begin{aligned} \hat{K}_{\text{ev}} : \mathcal{M} \times \text{SO}(3) & \rightarrow \mathcal{M} \times \text{SO}(3); \\ (U, V, \theta, \phi, \hat{h}) & \mapsto (V, U, \pi - \theta, \phi + \pi, \hat{h} \cdot \hat{g}_0^{-1}), \end{aligned} \quad (5.6)$$

where the dot denotes matrix multiplication in $\text{SO}(3)$.

The geon bundle for even n is not trivial. To see this, we view the geon bundle as the quotient $(\mathcal{M} \times \text{SO}(3)) / \{\text{Id}, \hat{K}_{\text{ev}}\}$. Suppose this bundle is trivial. Proceeding as in the discussion of Subsection 3.4 leading to (3.21), we see that there then exist a continuous 2π -periodic function $\gamma : \mathbb{R} \rightarrow \text{SO}(3)$ such that

$$\gamma(\phi + \pi) = \gamma(\phi) \cdot \hat{g}_0^{-1} \quad (5.7)$$

and the closed path $\gamma_0 : [0, 2\pi] \rightarrow \mathrm{SO}(3); \phi \mapsto \gamma(\phi)$ is contractible. We may assume without loss of generality that $\gamma(0) = \mathrm{Id} \in \mathrm{SO}(3)$. The condition (5.7) then implies that γ_0 is homotopic to the path $\gamma_1 : [0, 2\pi] \rightarrow \mathrm{SO}(3); \phi \mapsto \gamma_1(\phi)$, where

$$\gamma_1(\phi) := \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.8)$$

But as the lift of γ_1 to $\mathrm{SU}(2)$ is not closed, γ_1 is not contractible, and hence neither is γ_0 . This is a contradiction and implies that the assumed triviality of the geon bundle cannot hold.

6 Conclusions

We have shown that the static, spherically symmetric $\mathrm{SU}(n)$ black hole solutions of Künzle [20, 22] and the static, axially symmetric $\mathrm{SU}(2)$ black hole solutions of Kleihaus and Kunz [25, 26] admit topological geon quotients. These constructions extend the family of known non-abelian Einstein-Yang-Mills geon-type black holes from the static, spherically symmetric $\mathrm{SU}(2)$ geon-type black hole [12] to include geons with a more general Yang-Mills gauge group and to geons with less symmetry.

For Künzle's static, spherically symmetric $\mathrm{SU}(n)$ black holes with $n > 2$, we showed that a geon quotient generically requires an extension of the gauge group from $\mathrm{SU}(n)$ to $\mathbb{Z}_2 \ltimes \mathrm{SU}(n)$, where the nontrivial element of \mathbb{Z}_2 acts on $\mathrm{SU}(n)$ by complex conjugation. This means that the $\mathrm{SU}(n)$ charge conjugation must be treated as a gauge symmetry, rather than just as a global symmetry. This gauging is very similar to the $\mathrm{U}(1)$ charge conjugation gauging that is necessary for taking a geon quotient of the Reissner-Nordström black hole [12]. By contrast, static, spherically symmetric $\mathrm{SU}(2)$ black holes were known to admit a geon quotient without the need to gauge the $\mathrm{SU}(2)$ charge conjugation [12], and we showed that the same holds for the static, axially symmetric $\mathrm{SU}(2)$ black holes of Kleihaus and Kunz [25, 26].

In the cases where gauging the charge conjugation is not required, we showed that the geons built from Künzle's black holes have a trivial gauge bundle, whereas those built from the black holes of Kleihaus and Kunz have a trivial (respectively nontrivial) gauge bundle for odd (even) winding number of the gauge field configuration. We have not investigated whether this phenomenon reflects some deeper geometric property.

Our results on the axially symmetric solutions have a technical limitation in that the extension across the Killing horizon was C^0 but was not guaranteed to be differentiable. We suspect that this limitation is an artefact of a non-optimal coordinate choice and the results continue to hold within extensions of higher differentiability. It should be possible to examine this question with the techniques of Rácz and Wald [44, 45].

The topological geon black holes that we have found should provide an interesting arena for investigating the Hawking-Uhruh effect for quantum fields coupled to the

background Yang-Mills field. How does the geon's charge show up in the Hawking-Unruh effect, compared with the Hawking-Unruh effect on the conventional Kruskal-like extension? In particular, does the Hawking-Unruh effect feel the gauging of $SU(n)$ charge conjugation, as it does feel the gauging of $U(1)$ charge conjugation [18, 19]? When the charge conjugation is not gauged, does the Hawking-Unruh effect feel the triviality versus nontriviality of the geon's gauge bundle? A technically simple test field with which to address these questions might be a multiplet of charged scalars minimally coupled to the Yang-Mills field. A more interesting case might be a neutrino multiplet, for which the additional issue of inequivalent spin structures arises [15].

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